Linear Algebra Notes

Solving systems of equations:

Suppose we have n unknowns $x_1, x_2, \ldots, x_n$ and a system of n linear equations:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \end{aligned}$$

This system is equivalent to the following matrix equation:

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

We can write this system succinctly as $\boldsymbol{Ax} = \boldsymbol{b}$. We can solve this system of equations using the **augmented matrix**, which is the $n \times (n+1)$ matrix which is $\boldsymbol{A}$ concatenated with the vector $\boldsymbol{b}$:

$$\begin{pmatrix} a_{11} & a_{22} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & b_n \end{pmatrix}$$

Our Goal: Use elementary row operations (listed below) to transform the augmented matrix so that all the elements below the "diagonal" $(a_{11}, a_{22}, \ldots, a_{nn})$ are zero and all the elements on the diagonal are equal to one. Once our matrix is in this form, we can solve the system of equations.

Elementary Row Operations:

- interchange two rows,
- multiply a row by a constant,
- add a multiple of one row to another.

Example: Suppose we have the following system of linear equations:

$$\begin{aligned} x_1 + x_2 - 4x_3 &= 1 \\ 3x_1 + 2x_2 + 4x_3 &= -4 \\ 4x_1 + 3x_2 + 2x_3 &= -4 \end{aligned}$$

The augmented matrix associated to this is:

$$\begin{pmatrix} 1 & 1 & -4 & 1 \\ 3 & 2 & 4 & -4 \\ 4 & 3 & 2 & -4 \end{pmatrix}$$

Since the first entry of the first row is 1, there is no need to rescale the first row. So, our first step is to eliminate the first entry in the second and third rows. Suppose $\boldsymbol{r_1}, \boldsymbol{r_2}$ and $\boldsymbol{r_3}$ are the first second and third rows, respectively.

1. First, use the third elementary row operation to replace $\boldsymbol{r_2}$ with $\boldsymbol{r_2} - 3\boldsymbol{r_1}$.

$$\begin{pmatrix} 1 & 1 & -4 & 1 \\ 0 & -1 & 16 & -7 \\ 4 & 3 & 2 & -4 \end{pmatrix}$$

2. Next, use the third elementary row operation to replace $\boldsymbol{r_3}$ with $\boldsymbol{r_3} - 4\boldsymbol{r_1}$.

$$\begin{pmatrix} 1 & 1 & -4 & 1 \\ 0 & -1 & 16 & -7 \\ 0 & -1 & 18 & -8. \end{pmatrix}$$

3. Use the second elementary row operation to replace $\boldsymbol{r_2}$ with $-\boldsymbol{r_2}$.

$$\begin{pmatrix} 1 & 1 & -4 & 1 \\ 0 & 1 & -16 & 7 \\ 0 & -1 & 18 & -8. \end{pmatrix}$$

4. Next, use the third elementary row operation to replace $\boldsymbol{r_3}$ with $\boldsymbol{r_3} + \boldsymbol{r_2}$.

$$\begin{pmatrix} 1 & 1 & -4 & 1 \\ 0 & 1 & -16 & 7 \\ 0 & 0 & 2 & -1. \end{pmatrix}$$

5. Use the second elementary row operation to replace $\boldsymbol{r_3}$ with $\frac{1}{2}\boldsymbol{r_3}$.

$$\begin{pmatrix} 1 & 1 & -4 & 1 \\ 0 & 1 & -16 & 7 \\ 0 & 0 & 1 & -\frac{1}{2}. \end{pmatrix}$$

Now the matrix is in the right form, with 1's along the diagonal and 0's below. We can now solve our system of linear equations since the above augmented matrix is equivalent to:

$$\begin{aligned} x_1 + x_2 - 4x_3 &= 1 \\ x_2 - 16x_3 &= 7 \\ x_3 &= -1/2 \end{aligned}$$

From this, it easily follows that $x_3 = -\frac{1}{2}$, $x_2 = -1$, and $x_1 = 0$.

Important Note: Given matrix equation $\boldsymbol{Ax} = \boldsymbol{b}$, there are three possibilities:

- there is a unique solution,
- there is no solution, or
- there are infinitely many solutions.

When $\boldsymbol{b} = \mathbf{0}$, the zero vector, there will either be a unique solution or infinitely many solutions. The case when there is no solution doesn't happen since $\boldsymbol{x} = \mathbf{0}$ is always a solution.

You get a unique solution to $\boldsymbol{Ax} = \boldsymbol{b}$ exactly when $\det(\boldsymbol{A}) \neq 0$. The other two cases (no solution or infinitely many) occur when you have $\det(\boldsymbol{A}) = 0$. We obtained a unique solution for the example above. Let us verify that the determinant was nonzero.

Example: In our above example, we had the matrix $\boldsymbol{A}$ equal to:

$$\boldsymbol{A} = \begin{pmatrix} 1 & 1 & -4 \\ 3 & 2 & 4 \\ 4 & 3 & 2 \end{pmatrix}$$

Let's find the determinant of this matrix. To do this, we "expand" along the top row. By doing this, finding the determinant of a matrix becomes equivalent to finding the determinant of smaller matrices.

$$\begin{vmatrix} 1 & 1 & -4 \\ 3 & 2 & 4 \\ 4 & 3 & 2 \end{vmatrix} = 1\begin{vmatrix} 2 & 4 \\ 3 & 2 \end{vmatrix} - 1\begin{vmatrix} 3 & 4 \\ 4 & 2 \end{vmatrix} - 4\begin{vmatrix} 3 & 2 \\ 4 & 3 \end{vmatrix} = 1(4-12) - 1(6-16) - 4(9-8) = -2.$$

Since the determinant is nonzero, we know that the matrix equation $\boldsymbol{Ax} = \boldsymbol{b}$ will have a unique solution.

<u>Eigenvalues and eigenvectors:</u>

Given a matrix $\boldsymbol{A}$, we would like to find constants λ and vectors $\boldsymbol{x}$ (where $\boldsymbol{x} \neq \mathbf{0}$) so that

$$\boldsymbol{Ax} = \lambda\boldsymbol{x}.$$

When this happens, λ is called an **eigenvalue** of $\boldsymbol{A}$ and $\boldsymbol{x}$ is called the corresponding **eigenvector** of $\boldsymbol{A}$.

Letting $\mathbf{I}$ denote the identity matrix (1's along the diagonal and 0's elsewhere), we can rewrite the above formula as:

$$(\boldsymbol{A} - \lambda\boldsymbol{I})\boldsymbol{x} = 0.$$

Certainly, one solution to this equation would be the zero vector. However, we know that $\boldsymbol{x} \neq \mathbf{0}$. Therefore, if there is only a λ for which this formula could be true only if $\det(\boldsymbol{A}-\lambda\boldsymbol{I}) = 0$. The determinant of this matrix is an nth degree polynomial in λ. We can use this polynomial to find eigenvalues λ and then go back and find eigenvectors associated to λ. Let's see an example.

Example: Find the eigenvalues and eigenvectors for

$$\boldsymbol{A} = \begin{pmatrix} 2 & 4 \\ 3 & 2 \end{pmatrix}$$

First, we find the eigenvalues.

$$\boldsymbol{A} - \lambda\boldsymbol{I} = \begin{pmatrix} 2-\lambda & 4 \\ 3 & 2-\lambda \end{pmatrix}$$

Set the determinant of this matrix equal to 0.

$$\begin{vmatrix} 2-\lambda & 4 \\ 3 & 2-\lambda \end{vmatrix} = (2-\lambda)^2 - 12 = \lambda^2 - 4\lambda - 8 = 0.$$

We can solve, and find that the eigenvalues are $\lambda_1 = 2 + 2\sqrt{3}$ and $\lambda_2 = 2 - 2\sqrt{3}$.

To find the eigenvector for λ_1, we want to find $\boldsymbol{x_1}$ so that

$$(\boldsymbol{A} - \lambda_1\boldsymbol{I})\boldsymbol{x_1} = \mathbf{0}.$$

Plugging in our value for λ_1, we have:

$$\begin{pmatrix} -2\sqrt{3} & 4 \\ 3 & -2\sqrt{3} \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

In general, we can use the augmented matrix to solve this system of equations to find $\boldsymbol{x_1}$. This is a simple system of equations which we can solve quickly. We find that $y_1 = \frac{\sqrt{3}}{2}x_1$. (Since the $\det(\boldsymbol{A}-\lambda\boldsymbol{I}) = 0$, there are infinitely many solutions!) So in fact there are infinitely many eigenvectors, but they are all constant multiples of each other, so we can pick one that we like. If I choose $x_1 = \sqrt{3}$, the I get the eigenvector $\boldsymbol{x_1}$ corresponding to the eigenvalue $\lambda_1 = 2 + 2\sqrt{3}$:

$$\boldsymbol{x_1} = \begin{pmatrix} \sqrt{3} \\ 3/2 \end{pmatrix}.$$

We can similarly find an eigenvalue for $\lambda_2 = 2 - 2\sqrt{3}$:

$$\begin{pmatrix} 2\sqrt{3}) & 4 \\ 3 & 2\sqrt{3} \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

From this, we deduce that $y_2 = -\frac{\sqrt{3}}{2}x_2$. If I choose $x_2 = \sqrt{3}$, the I get the eigenvector $\boldsymbol{x_2}$ corresponding to the eigenvalue $\lambda_2 = 2 - 2\sqrt{3}$:

$$\boldsymbol{x_2} = \begin{pmatrix} \sqrt{3} \\ -3/2 \end{pmatrix}.$$